Every path-connected space is connected

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Proof idea: The proof is by contradiction. We take a path-connected space and assume it's not connected. From this we can easily show that $[0, 1]$ is not connected, which is a contradiction.

To be able to prove the main result, we first prove a simple lemma:

Lemma: Let A be a non-empty, closed and bounded set in $\mathbb{R}$. Then it contains its supremum and infimum.

Proof: We prove the supremum claim; the proof of the infimum claim is similar. Assume that $s = \sup(A) \notin A$. Since A is closed, it contains all its boundary points. It follows that s is contained in the exterior of A, and thus there is an open neighbourhood U of s for which $U \cap A = \emptyset$. Finally we can find an interval $]s - \varepsilon, s + \varepsilon[\subset U$, and we've found a upper bound for A that is smaller than s, which is a contradiction. Thus $\sup(A) \in A$. □

Now we can move on to the main proof:

Proof of main theorem: Let X be a path-connected topological space, and assume that it is not connected. Then we have two open non-empty sets $A \subset X$ and $B \subset X$ for which $X = A \cup B$. Take points $a \in A$ and $b \in B$ and connect them with a path $f : [0, 1] \to X$. Now $f^{-1}A$ and $f^{-1}B$ are two disjoint open non-empty sets in $[0, 1]$ whose union is $[0, 1]$, and thus $[0, 1]$ is not connected.

Since $f^{-1}A$ is open in $[0, 1]$, and its complement in $[0, 1]$ is $f^{-1}B$, we know that $f^{-1}B$ is closed in $[0, 1]$, and by the same argument $f^{-1}A$ is also closed in $[0, 1]$. Since $[0, 1]$ is closed in $\mathbb{R}$, $f^{-1}A$ and $f^{-1}B$ are closed in $\mathbb{R}$. Thus $f^{-1}A$ and $f^{-1}B$ are closed and bounded non-empty subsets of $\mathbb{R}$, which means that they contain their infimum and supremum by the lemma we proved earlier.

Because $[0, 1] = f^{-1}A \cup f^{-1}B$, one of the sets must have infimum 0. We assume that $\inf(f^{-1}A) = 0$. We also know that $f^{-1}A \cap f^{-1}B = \emptyset$, so $b = \inf(f^{-1}B)$ must be positive. Because $f^{-1}B$ is open in $[0, 1]$, the point b must have an open neighbourhood $U \subset [0, 1]$ which is contained in $f^{-1}B$. Thus we have an open set $V \subset \mathbb{R}$ for which $U = V \cap \mathbb{R}$, which means we can find an open interval $]b - \varepsilon\ ,\ b + \varepsilon[\subset\ f^{-1}B$ for some $\varepsilon > 0$, $\varepsilon < b$. It follows that $f^{-1}B$ contains a number $0 < b - \frac{\varepsilon}{2} < b$, which is a contradiction, since $b = \inf(f^{-1}B)$. Thus X must be connected. □