Two counterexamples in algebra

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Abstract

This text presents two interesting counterexamples in algebra concerning isomorphisms between quotient groups of isomorphic groups and commutative operators.

I first stumbled upon the example concerning isomorphisms during a course in Galois theory. The second example about commutative operators was presented to me during a course in finite-dimensional linear algebra.

Remark. In this text we denote isomorphisms and isomorphic groups by $f : G \cong H$.

Example. Isomorphisms between quotient groups of isomorphic groups. Assume that $f : G_1 \to G_2$ is a group isomorphism. If H_1 is a normal subgroup of G_1, then fH_1 is a normal subgroup of G_2 and we know that $G_1/H_1 \cong G_2/fH_1$.

Let us now again assume that $G_1 \cong G_2$, and that we have normal subgroups H_1 of G_1 and H_2 of G_2 for which it holds that $H_1 \cong H_2$. Must it then necessarily hold that $G_1/H_1 \cong G_2/H_2$? The answer is no. Let $G_1 = G_2 = \mathbb{Z}$, and let $H_1 = 2\mathbb{Z}$ and $H_2 = 3\mathbb{Z}$. Now $G_1 \cong G_2$ and $H_1 \cong \mathbb{Z} \cong H_2$, but for the quotient groups we obtain

$$G_1/H_1 = \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2 \ncong \mathbb{Z}_3 \cong \mathbb{Z}/3\mathbb{Z} = G_2/H_2.$$

The problem here is that the isomorphism f from G_1 and G_2 is the identity function on $\mathbb{Z}$, but the isomorphism g from H_1 and H_2 is **not** the identity function, and is thus not a restriction of f to H_1. Thus we may not use the reasoning that we used in the beginning of the example.

We also notice that even if G_1 and G_2 are finite groups, the result must not necessarily hold. To see this, let $G_1 = G_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ (the direct sum of $\mathbb{Z}_2$ and $\mathbb{Z}_4$), and let $H_1 = \{(0,0),(1,0)\}$ and $H_2 = \{(0,0),(0,2)\}$. Now again obviously it holds that $G_1 \cong G_2$, and we also see that $H_1 \cong \mathbb{Z}_2 \cong H_2$.

To determine the quotient group G_1/H_1, we notice that for every $x \in \mathbb{Z}_4$ it holds that $(0,x) + H_1 = (1,x) + H_1$, since $(0,x) + (1,0) = (1,x)$, and $(1,0) \in H_1$. Thus, it follows that

$$\begin{aligned} G_1/H_1 &= \{g + H_1 \mid g \in G\} \\ &= \{(0,0) + \{(0,0),(1,0)\}, (0,1) + \{(0,0),(1,0)\}, \\ &\quad (0,2) + \{(0,0),(1,0)\}, (0,3) + \{(0,0),(1,0)\}\} \\ &\cong \mathbb{Z}_4. \end{aligned}$$

On the other hand, we see that since $(0,2) \in H_2$, we know that

$$\begin{aligned}(0,0)+H_2 &= (0,2)+H_2,\\ (1,0)+H_2 &= (1,2)+H_2,\\ (1,1)+H_2 &= (1,3)+H_2,\\ (0,1)+H_2 &= (0,3)+H_2.\end{aligned}$$

Thus every element in G_2/H_2 is its own inverse, since

$$\begin{aligned}((1,0)+(1,0))+H_2 &= (0,0)+H_2,\\ ((1,1)+(1,1))+H_2 &= (0,2)+H_2 = (0,0)+H_2,\\ ((0,1)+(0,1))+H_2 &= (0,2)+H_2 = (0,0)+H_2.\end{aligned}$$

This means that $G_2/H_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, i.e. the Klein four-group, and since the Klein four-group isn't cyclic but $\mathbb{Z}_4$ is, we know that $G_1/H_1 \ncong G_2/H_2$.

This example shows that it's important to not immediately consider isomorphic groups to be the same in all ways, since we just showed that isomorphic groups can have a very different relation to some outer structure. It's also good to remember that when talking about quotient groups, one should always exactly specify what normal subgroup is used, since isomorphic subgroups don't necessarily create isomorphic quotient groups. This also means that quotient group notations like

$$\frac{\mathbb{Z}_2 \oplus \mathbb{Z}_4}{\mathbb{Z}_2}$$

may easily be misunderstood, since the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ has more than one subgroup that is isomorphic to $\mathbb{Z}_2$.

Example. Commutativity does not imply associativity.
In algebra, Abelian groups are a special case of groups, since in the usual definition of a group, we only require that there is a neutral element, that every element has an inverse and that the operator is associative. Since commutativity is often used only in the context of groups, one might think that commutativity is in some way a stronger property than associativity, i.e. that if an operator is commutative, it must also be associative. This is, however, not the case. A simple example is the operator $\oplus : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, $a \oplus b = (a+b)/2$. This operator is obviously commutative, but it is not associative, since

$$\begin{aligned}(4\oplus 4)\oplus 8 &= \frac{4+4}{2}\oplus 8 = 4\oplus 8 = \frac{4+8}{2} = 6, \text{but}\\ 4\oplus(4\oplus 8) &= 4\oplus\frac{4+8}{2} = 4\oplus 6 = \frac{4+6}{2} = 5.\end{aligned}$$